

APPROXIMATING REALS BY RATIONALS OF THE FORM a/b^2 .

OLEG KARPENKOV

ABSTRACT. In this note we formulate some questions in the study of approximations of reals by rationals of the form a/b^2 arising in theory of Shrödinger equations. We hope to attract attention of specialists to this natural subject of number theory.

1. INTRODUCTION

Some background. In this note we formulate some questions in the study of approximations of reals by rationals of the form a/b^2 arising in theory of Shrödinger equations (see [3] and [5] for further information). We hope to attract attention of specialists to this natural subject of number theory. Good references to theory of approximations by arbitrary algebraic numbers are for instance [11], [8] and [9], especially for approximations by quadratic irrationals, see [4]. A metric approach to the study (in a more general situation) was proposed by [7] and further developed by M. Weber in [10], V. Beresnevich, M. Dodson, S. Kristensen, and J. Levesley in [1] and other works. This approach is a good test of the proposed problems, nevertheless it does not give the answers. Some upper bound estimates were made by A. Zaharescu in [12].

2. QUESTIONS

We start with formulation of one of the main results in classical approximation theory of reals by rationals (see [6] for the proofs).

Theorem 1. I. *For any reals α and $c \geq 1/\sqrt{5}$ there exists an infinite number of integer solutions (a, b) , $b > 0$ for the following inequality*

$$\left| \alpha - \frac{a}{b} \right| < \frac{c}{b^2}.$$

II. *Let α be the Golden Ratio (i. e. $\alpha = (\sqrt{5} + 1)/2$). Then for any $c < 1/\sqrt{5}$ the inequality of item I. has only finitely many solutions.* \square

Similar results for the approximations by rationals of the form a/b^2 are not known. One of the reasons of that is the following: lattice geometry of continued fractions corresponding to the approximations by rationals can not be naturally expanded to the case of rationals a/b^2 .

Problem 1. Find a good generalization for geometry of numbers to the case of a/b^2 -approximations.

Let us give some known estimates for the case of a/b^2 -approximations. The lower estimate seems to be quite precise.

Date: 30 October 2006.

Key words and phrases. Approximations of reals by rationals.

Partially supported by NWO-RFBR 047.011.2004.026 (RFBR 05-02-89000-NWO_a) grant, by RFBR SS-1972.2003.1 grant, by RFBR 05-01-02805-CNRS_La grant, and by RFBR grant 05-01-01012a.

Theorem 2. *For any positive ε and for any real α there exist a positive constant $c = c(\varepsilon)$, such that the following inequality does not have integer solutions:*

$$\left| \alpha - \frac{a}{b^2} \right| < \frac{c}{b^3 \ln^{1+\varepsilon}(b)}.$$

The proof of a more general statement is given by I. Borosh and A. S. Fraenkel in [2]. We suppose that the logarithm in the formula can be eliminated.

All known proofs of previous theorem are general, and do not give the examples of badly approximable reals, like it was $(\sqrt{5}+1)/2$ for the case of approximations by a/b . So the following problem is actual here.

Problem 2. Find any particular example of α that satisfies the condition of Theorem 2.

The following estimate for the upper bound case is known.

Theorem 3. A. Zaharescu [12]. *For any real α and any positive $\theta < 2/3$ there exists infinitely many solutions of the following inequality:*

$$\left| \alpha - \frac{a}{b^2} \right| < \frac{1}{b^{2+\theta} \xi(b)}.$$

As one can see there is a gap between upper and lower estimates for the badly approximable reals by a/b^2 -rationals. The results [2] for almost all reals and numerical experiments support the following classical conjecture.

Conjecture 3. For any real α there exists a constant $c(\alpha)$ such that the inequality:

$$\left| \alpha - \frac{a}{b^2} \right| < \frac{c(\alpha)}{b^3}$$

has infinitely many integer solutions (a, b) .

We conclude this note with the following problem which is supposed to be close to the subject.

Problem 4. Find the estimates for the upper and lower bounds of the “best” approximations of reals by rationals of the form z/p , where z is integer, and p is prime or unity.

Acknowledgement. The author is grateful to E. Séré, W. Craig, H. W. Lenstra, J.-H. Evertse, S. Kristensen, and Nigel Watt, for help with collecting the information and comments, and Mathematisch Instituut of Universiteit Leiden for the hospitality and excellent working conditions.

REFERENCES

- [1] V. Beresnevich, M. Dodson, S. Kristensen and J. Levesley, *An inhomogeneous wave equation and non-linear Diophantine approximation*, preprint, March 2006, 18 p.
- [2] I. Borosh, A. S. Fraenkel, *A generalization of Jarník’s theorem on Diophantine approximations*, Indag. Math. 34 (1972), pp. 193–201.
- [3] W. Craig, *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, Panoramas et synthèses, 9, SMF, Paris 2000.
- [4] H. Davenport, W. M. Schmidt, *Approximation to real numbers by quadratic irrationals*, Acta Arith., 13(1967/1968), pp. 169–176.
- [5] S. Kristensen, *Diophantine approximation and the solubility of the Schrodinger equation*, Phys. Lett. A, 314(1-2), 2003, pp. 15-18.
- [6] A. Ya. Khinchin, *Continued fractions*, Moscow, FIZMATGIZ, 3. ed., (1961); English translation University of Chicago Press, 1961.
- [7] W. M. Schmidt, *Metrical theorems on fractional parts of sequences*, Trans. Amer. Math. Soc., 110(1964), pp. 493–518.

- [8] V. G. Sprindžuk, *A proof of Mahler's conjecture on the measure of the set of S -numbers* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 29(1965), pp. 379–436.
 - [9] K. I. Tishchenko, *On approximation to real numbers by algebraic numbers*, Acta Arith., 94(2000), no. 1, pp. 1–24.
 - [10] M. Weber, *Some examples of application of the metric entropy method*, Acta Math. Hungar. 105(2004), no. 1-2, pp. 39–83.
 - [11] E. Wirsing, *Approximation mit algebraischen Zahlen beschränkten Grades* (German), J. Reine Angew. Math., 206(1960), pp. 67–77.
 - [12] , A. Zaharescu, *Small values of $n^2\alpha \pmod{1}$* , Invent. Math., 121(1995), no. 2, pp. 379–388.
- E-mail address*, Oleg Karpenkov: `karpenk@mccme.ru`

MATHEMATISCH INSTITUUT, UNIVERSITEIT LEIDEN, P.O. BOX 9512, 2300 RA LEIDEN, THE NETHERLANDS